An Improved Diversity Mechanism for Solving Constrained Optimization Problems Using a Multimembered Evolution Strategy

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Abstract. This paper presents an improved version of a simple evolution strategy (SES) to solve global nonlinear optimization problems. As its previous version, the approach does not require the use of a penalty function, it does not require the definition by the user of any extra parameter (besides those used with an evolution strategy), and it uses some simple selection criteria to guide the process to the feasible region of the search space. Unlike its predecessor, this new version uses a multimembered Evolution Strategy (ES) and an improved diversity mechanism based on allowing infeasible solutions close to the feasible region to remain in the population. This new version was validated using a well-known set of test functions. The results obtained are very competitive when comparing the proposed approach against the previous version and other approaches representative of the state-of-the-art in constrained evolutionary optimization. Moreover, its computational cost (measured in terms of the number of fitness function evaluations) is lower than the cost required by the other techniques compared.

1 Introduction

Evolutionary algorithms (EAs) have been successfully used to solve different types of optimization problems [1]. However, in their original form, they lack an explicit mechanism to handle the constraints of a problem. This has motivated the development of a considerable number of approaches to incorporate constraints into the fitness function of an EA [2,3]. Particularly, in this paper we are interested in the general nonlinear programming problem in which we want to: Find $\boldsymbol{x}$ which optimizes $f(\boldsymbol{x})$ subject to: $g_i(\boldsymbol{x}) \leq 0, \;\; i = 1, \ldots, n$ $h_j(\boldsymbol{x}) = 0, \;\; j = 1, \ldots, p$ where $\boldsymbol{x}$ is the vector of decision variables $\boldsymbol{x} = [x_1, x_2, \ldots, x_r]^T$, n is the number of inequality constraints and p is the number of equality constraints (in both cases, constraints could be linear or nonlinear).

The most common approach adopted to deal with constrained search spaces is the use of penalty functions [4]. When using a penalty function, the amount of constraint violation is used to punish or "penalize" an infeasible solution so that feasible solutions

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are favored by the selection process. Nonetheless, the main drawback of penalty functions is that they require a careful fine tuning of the penalty factors that accurately estimates the degree of penalization to be applied so that we can approach efficiently the feasible region [3].

The algorithm presented in this paper is an improved version of two previous approaches. The first version [5] was based on a $(\mu + 1)$ Evolution Strategy coupled with three simple selection criteria based on feasibility to guide the search to the feasible region of the search space. A second version of the approach was proposed in [6] but now using a $(1 + \lambda)$-ES and adding a diversity mechanism which consisted of allowing solutions with a good value of the objective function to remain as a new starting point in the next generation of the search, regardless of feasibility. The version presented in this paper still uses the self-adaptive mutation mechanism of an ES, but we now adopt a multimembered $(\mu + \lambda)$-ES to explore constrained search spaces. This mechanism is combined with the same three simple selection criteria used before to guide the search towards the global optima of constrained optimization problems [6]. However, we now add an improved diversity mechanism which, although simple, provides a significant improvement in terms of performance. The idea of this mechanism is to allow the individual with both the lowest amount of constraint violation and the best value of the objective function to be selected for the next generation. This solution can be chosen with a 50% of probability either from the parents or from the offspring population. With the combination of the above elements, the algorithm first focuses on reaching the feasible region of the search space. After that, it is capable of moving over the feasible region as to reach the global optimum. The infeasible solutions that remain in the population are then used to sample points in the boundaries between the feasible and the infeasible regions. Thus, the main focus of this paper is to show how a multimembered ES coupled with the previously described diversity mechanism, has a highly competitive performance in constrained problems when compared with respect to algorithms representative of the state-of-the-art in the area.

This paper is organized as follows: In Section 2 a description of previous approaches based on similar ideas to our own is provided. Section 3 includes the description of the diversity mechanism that we propose. Then, in Section 4, we present the results obtained and also a comparison against the previous version and state-of-the-art algorithms. Such results are discussed in Section 5. Finally, in Section 6 we provide some conclusions and possible paths for future research.

2 Related Work

The hypothesis that originated this work is the following: (1) The self-adaptation mechanism of an ES helps to sample the search space well enough as to reach the feasible region reasonably fast and (2) the simple addition of simple selection criteria based on feasibility to an ES should be enough to guide the search in such a way that the global optimum can be approached efficiently.

The three simple selection criteria used are the following:

1. Between 2 feasible solutions, the one with the highest fitness value wins (assuming a maximization problem/task).

2. If one solution is feasible and the other one is infeasible, the feasible solution wins.
3. If both solutions are infeasible, the one with the lowest sum of constraint violation is preferred.

The use of these criteria has been explored by other authors. Jiménez and Verdegay [7] proposed an approach similar to a min-max formulation used in multiobjective optimization combined with tournament selection. The rules used by them are similar to those adopted in this work. However, Jiménez and Verdegay's approach lacks an explicit mechanism to avoid the premature convergence produced by the random sampling of the feasible region because their approach is guided by the first feasible solution found. Deb [8] used the same tournament rules previously indicated in his approach. However, Deb proposed to use niching as a diversity mechanism, which introduces some extra computational time (niching has time-complexity $O(N^2)$). In Deb's approach, feasible solutions are always considered better than infeasible ones. This contradicts the idea of allowing infeasible individuals to remain in the population. Therefore, this approach will have difficulties in problems in which the global optimum lies on the boundary between the feasible and the infeasible regions.

Motivated by the fact that some of the most recent and competitive approaches to incorporate constraints into an EA use an ES (see for example [9,10]), we proposed [5] a Simple $(\mu + 1)$ Evolution Strategy (SES) to solve constrained optimization problems in which one child created from μ mutations of the current solution competes against it and the better one is selected as the new current solution. This approach is based on the two mechanisms previously indicated.

However, the approach in [5] used to get trapped in local optimum solutions. Thus, in order to improve the quality and robustness of the results, a diversity mechanism was added in [6]. In this case, a $(1+\lambda)$-ES was adopted and the diversity mechanism consisted on allowing solutions with a good value of the objective function to remain as a new starting point for the search at each generation, regardless of feasibility. Additionally, we introduced a self-adaptive parameter called Selection Ratio (S_r), which refers to the percentage of selections that will be performed in a deterministic way (as used in the first version of the SES [5] where the child replaces the current solution based on the three selection criteria previously indicated). In the remaining $1 - S_r$ selections, there were two choices: (1) either the individual (out of the λ) with the best value of the objective function would replace the current solution (regardless of its feasibility) or (2) the best parent (based on the three selection criteria) would replace the current solution. Both options are given a 50% probability each. The results improved, but for some test problems no feasible solutions could be found and for other functions the statistical results did not show enough robustness.

3 The New Diversity Mechanism

The two previous versions of the algorithm [5,6] are based on a single-membered ES and they lack the explorative power to sample large search spaces. Thus, we decided to re-evaluate the use of a $(\mu + \lambda)$-ES to solve this limitation, but in this case, improving the diversity mechanism implemented in the second version of our approach [6] and eliminating the use of the self-adaptive S_r parameter. The new version of the SES is based on the same concepts that its predecessors as discussed before.

The detailed features of the improved diversity mechanism are the following: At each generation, we allow the infeasible individual with the best value of the objective function and with the lowest amount of constraint violation to survive for the next generation. We call this solution the best infeasible solution. In fact, there are two best infeasible solutions at each generation, one from the μ parents and one from the λ offspring. Either of them can be chosen with a 50% of probability. With 0.03 probability, the selection process will choose the best infeasible individual with equal probability to be the best infeasible parent or the best infeasible offspring.

Therefore, the same best infeasible solution can be copied more than once into the next population. However, this is a desired behavior because a few copies of this solution will allow its recombination with several solutions in the population, specially with feasible ones. Recombining feasible solutions with infeasible solutions in promising areas (based on the good value of the objective function) and close to the boundary of the feasible region will allow the ES to reach global optimum solutions located in the boundary of the feasible region of the search space (which are known to be the most difficult solutions to reach). See Figure 1.

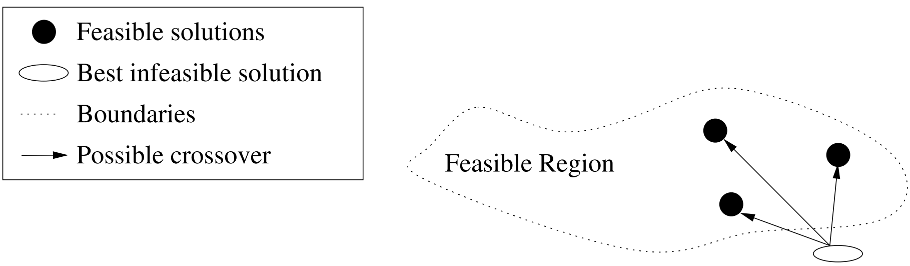

Fig. 1. Diagram that illustrates the idea of searching the boundaries with the new diversity mechanism proposed in this paper.

When the selection process occurs, the best individuals among the parents and offspring are selected based on the three selection criteria previously indicated. The selection process will pick feasible solutions with a better value of the objective function first, followed by infeasible solutions with a lower constraint violation. However, 3 times from every 100 picks, the best infeasible solution (from either the parents or the offspring population with a 50% of probability each) the best infeasible solution is copied in the population for the next generation. The pseudocode is listed in Figure 2. We chose the value of 3 based on the previous version [6] which used a population of just 3 offspring. With this low number of solutions, the approach provided good results.

```
function selection()
    For i=1 to μ Do
        If flip(0.97)
            Select the best individual based on the selection criteria from the union
            of the parents and offspring population, add it to the population for the
            next generation and delete it from this union.
        Else
            If flip(0.5)
                Select the best infeasible individual from the parents population and
                add it to the population for the
                next generation.
            Else
                Select the best infeasible individual from the offspring population and
                add it to the population for
                the next generation.
            End If
        End If
    End For
End
```

Fig. 2. Pseudocode of the selection procedure with the diversity mechanism incorporated. $flip(P)$ is a function that returns TRUE with probability P

4 Experiments and Results

To evaluate the performance of the proposed approach we used the 13 test functions described in [9]. The test functions chosen contain characteristics that are representative of what can be considered "difficult" global optimization problems for an evolutionary algorithm. Their expressions are provided in the Appendix at the end of this paper.

To get an estimate of the ratio between the feasible region and the entire search space for these problems, a ρ metric (as suggested by Michalewicz and Schoenauer [2]) was computed using the following expression: $\rho = |F|/|S|$ where $|F|$ is the number of feasible solutions and $|S|$ is the total number of solutions randomly generated. In this work, $S = 1,000,000$ random solutions.

The different values of ρ for each of the functions chosen are shown in Table 4, where n is the number of decision variables, LI is the number of linear inequalities, NI the number of nonlinear inequalities, LE is the number of linear equalities and NE is the number of nonlinear equalities. We performed 30 independent runs for each test function. The learning rates values were calculated using the formulas proposed by Schwefel [12] (where n is the number of decision variables of the problem): $\tau = (\sqrt{2\sqrt{n}})^{-1}$ and $\tau' = (\sqrt{2n})^{-1}$. In order to favor finer movements in the search space (as we observed in the previous versions of the approach where only one sigma value was used and when it had values close to zero the improvements of the result increased) we decided to experiment with just a percentage of the quantity obtained by the formula proposed by Schwefel [12]. We initialized the sigma values for all the individuals in the initial

Table 1. Statistical results obtained by our SES for the 13 test functions with 30 independent runs. A result in **boldface** means global optimum solution found.

	Statistical Results of the New SES with the Improved Diversity Mechanism					
Problem	Optimal	Best	Mean	Median	Worst	St. Dev.
g01	−15.00	**−15.00**	**−15.00**	**−15.00**	**−15.00**	0
g02	0.803619	0.803601	0.785238	0.792549	0.751322	1.67E-2
g03	1.00	**1.00**	**1.00**	**1.00**	**1.00**	2.09E-4
g04	−30665.539	**−30665.539**	**−30665.539**	**−30665.539**	**−30665.539**	0
g05	5126.498	5126.599	5174.492	5160.198	5304.167	50.05E+0
g06	−6961.814	**−6961.814**	−6961.284	−6961.814	−6952.482	1.85E+0
g07	24.306	24.327	24.475	24.426	24.843	1.32E-1
g08	0.095825	**0.095825**	**0.095825**	**0.095825**	**0.095825**	0
g09	680.630	680.632	680.643	680.642	680.719	1.55E-2
g10	7049.25	7051.90	7253.05	7253.60	7638.37	136.0E+0
g11	0.75	**0.75**	0.75	0.75	0.75	1.52E-4
g12	1.00	**1.00**	1.00	1.00	1.00	0
g13	0.053950	0.053986	0.166385	0.061873	0.468294	1.76E-1

Table 2. Comparison of results between the new SES and the old one proposed in [6]. "-" means no feasible solutions were found. A result in **boldface** means a better value obtained by our new approach.

		Best Result		Mean Result		Worst Result	
Problem	Optimal	NEW SES	OLD	NEW SES	OLD	NEW SES	OLD
g01	−15.00	−15.00	−15.00	−15.00	−15.00	−15.00	−15.00
g02	0.803619	**0.803601**	0.803569	**0.785238**	0.769612	**0.751322**	0.702322
g03	1.00	1.00	1.00	1.00	1.00	1.00	1.00
g04	−30665.539	−30665.539	−30665.539	−30665.539	−30665.539	−30665.539	−30665.539
g05	5126.498	**5126.599**	−	**5174.492**	−	**5304.167**	−
g06	−6961.814	−6961.814	−6961.814	−6961.284	−6961.814	−6952.482	−6961.814
g07	24.306	24.327	24.314	24.475	24.419	24.843	24.561
g08	0.095825	0.095825	0.095825	**0.095825**	0.095784	**0.095825**	0.095473
g09	680.630	**680.632**	680.669	**680.643**	680.810	**680.719**	681.199
g10	7049.25	**7051.90**	7057.04	**7253.05**	10771.42	**7638.37**	16375.27
g11	0.75	0.75	0.75	0.75	0.75	0.75	**0.76**
g12	1.00	1.00	1.00	1.00	1.00	1.00	1.00
g13	0.053950	0.053986	0.053964	**0.166385**	0.264135	**0.468294**	0.544346

Table 3. Comparison of the new version of the SES with respect to the Homomorphous Maps (HM) [11]. "-" means no feasible solutions were found. A result in **boldface** means a better value obtained by our new approach.

		Best Result		Mean Result		Worst Result	
Problem	Optimal	New SES	HM	New SES	HM	New SES	HM
g01	−15.00	**−15.00**	−14.7886	**−15.00**	−14.7082	**−15.00**	−14.6154
g02	0.803619	**0.803601**	0.79953	0.785238	0.79671	0.751322	0.79119
g03	1.00	**1.00**	0.9997	**1.00**	0.9989	**1.00**	0.9978
g04	−30665.539	**−30665.539**	−30664.5	**−30665.539**	−30655.3	**−30665.539**	−30645.9
g05	5126.498	**5126.599**	−	**5174.492**	−	**5304.167**	−
g06	−6961.814	**−6961.814**	−6952.1	**−6961.284**	−6342.6	**−6952.482**	−5473.9
g07	24.306	**24.327**	24.620	**24.475**	24.826	**24.843**	25.069
g08	0.095825	0.095825	0.0958250	**0.095825**	0.0891568	**0.095825**	0.0291438
g09	680.63	**680.632**	680.91	**680.643**	681.16	**680.719**	683.18
g10	7049.25	**7051.90**	7147.9	**7253.05**	8163.6	**7638.37**	9659.3
g11	0.75	0.75	0.75	0.75	0.75	0.75	0.75
g12	1.00	**1.00**	0.999999	**1.00**	0.999134	**1.00**	0.991950
g13	0.053950	0.053986	NA	0.166385	NA	0.468294	NA

Table 4. Values of ρ for the 13 test problems chosen.

Problem	n	Function	ρ	LI	NI	LE	NE
g01	13	quadratic	0.0003%	9	0	0	0
g02	20	nonlinear	99.9973%	1	1	0	0
g03	10	nonlinear	0.0026%	0	0	0	1
g04	5	quadratic	27.0079%	0	6	0	0
g05	4	nonlinear	0.0000%	2	0	0	3
g06	2	nonlinear	0.0057%	0	2	0	0
g07	10	quadratic	0.0000%	3	5	0	0
g08	2	nonlinear	0.8581%	0	2	0	0
g09	7	nonlinear	0.5199%	0	4	0	0
g10	8	linear	0.0020%	3	3	0	0
g11	2	quadratic	0.0973%	0	0	0	1
g12	3	quadratic	4.7697%	0	9^3	0	0
g13	5	nonlinear	0.0000%	0	0	1	2

population with only a 40% of the value obtained by the following formula (where n is the number of decision variables): $\sigma_i(0) = 0.4 \times (\Delta x_i/\sqrt{n})$ where Δx_i is approximated with the expression (suggested in [9]), $\Delta x_i \approx x_i^u - x_i^l$, where $x_i^u - x_i^l$ are the upper and lower limits of the decision variable i. For the experiments we used the following parameters: $(100+300)$-ES, number of generations = 800, number of objective function evaluations = $240,000$.

To increase the exploitation feature of the global crossover operator we combine discrete and intermediate crossover. Each gene in the chromosome can be processed with any of these two crossover operators with a 50% of probability. This operator is applied to both, strategy parameters (sigma values) and decision variables of the problem. Note that we do not use correlated mutation. To deal with equality constraints, a parameterless dynamic mechanism originally proposed in ASCHEA [10] and used in [5] and in [6] is adopted. The tolerance value ϵ is decreased with respect to the current generation using the following expression: $\epsilon_j(t+1) = \epsilon_j(t)/1.00195$. The initial ϵ_0 was set to 0.001. For problem g13, ϵ_0 was set to 3.0 and, in consequence, the factor to decrease the tolerance value was modified to $\epsilon_j(t+1) = \epsilon_j(t)/1.0145$. Also, for problems $g03$ and $g13$ the initial stepsize required a more dramatic decrease of the stepsize. They were defined as 0.01 (just a 5% instead of the 40%) for $g03$ and 0.05 (a 2.5% instead of the 40%) for g13. These two test functions seem to provide better results with very smooth movements. It is important to note that these two problems share the following features: moderately high dimensionality (five or more decision variables), nonlinear objective function, one or more equality constraints, and moderate size of the search space (based on the range of the decision variables). These common features suggest that for this type of problem, finer movements provide a better sampling of the search space using an evolution strategy.

The statistical results of this new version of the SES with the improved diversity mechanism are summarized in Table 1. The comparison of the improved version against the previous one [6] is presented in Table 2. We compared our approach against the previous version of the SES [6] in Table 2 and against three state-of-the-art approaches: the Homomorphous Maps (HM) [11] in Table 3, Stochastic Ranking (SR) [9] in Table 5 and the Adaptive Segregational Constraint Handling Evolutionary Algorithm (ASCHEA) [10] in Table 6.

The Homomorphous Maps performs a homomorphous mapping between an n-dimensional cube and the feasible search region (either convex or non-convex). The main idea of this approach is to transform the original problem into another (topologically equivalent) function that is easier to optimize by the EA. Both, the Stochastic Ranking and ASCHEA are based on a penalty function approach. SR sorts the individuals in the population in order to assign them a rank value. However, based on the value of a user-defined parameter, the comparison between two adjacent solutions will be performed using only the objective function. The remaining comparisons will be performed using only the penalty value (the sum of constraint violation). ASCHEA uses three combined mechanisms: (1) an adaptive penalty function, (2) a constraint-driven recombination that forces to select a feasible individual to recombine with an infeasible one and (3) a segregational selection based on feasibility which maintains a balance between feasible and infeasible solutions in the population. ASCHEA also requires a niching mechanism to improve the diversity in the population. Each mechanism requires the definition by the user of extra parameters.

Table 5. Comparison of our new version of the SES with respect to Stochastic Ranking (SR) [9]. A result in **boldface** means a better value obtained by our new approach.

		Best Result		Mean Result		Worst Result	
Problem	**Optimal**	**New SES**	**SR**	**New SES**	**SR**	**New SES**	**SR**
g01	−15.00	−15.00	−15.000	−15.00	−15.000	−15.00	−15.000
g02	0.803619	**0.803601**	0.803515	**0.785238**	0.781975	**0.751322**	0.726288
g03	1.00	1.00	1.000	1.00	1.000	1.00	1.000
g04	−30665.539	−30665.539	−30665.539	−30665.539	−30665.539	−30665.539	−30665.539
g05	5126.498	5126.599	5126.497	5174.492	5128.881	5304.165	5142.472
g06	−6961.814	−6961.814	−6961.814	**−6961.284**	−6875.940	**−6952.482**	−6350.262
g07	24.306	24.327	24.307	24.475	24.374	24.843	24.642
g08	0.095825	0.095825	0.095825	0.095825	0.095825	0.095825	0.095825
g09	680.63	680.632	680.630	**680.643**	680.656	**680.719**	680.763
g10	7049.25	**7051.90**	7054.316	**7253.05**	7559.192	**7638.37**	8835.655
g11	0.75	0.75	0.750	0.75	0.750	0.75	0.750
g12	1.00	1.00	1.00	1.00	1.00	1.00	1.00
g13	0.053950	0.053986	0.053957	0.166385	0.057006	0.468294	0.216915

5 Discussion of Results

As described in Table 1, our approach was able to find the global optimum in seven test functions (g01, g03, g04, g06, g08, g11 and g12) and it found solutions very close to the global optimum in the remaining six (g02, g05, g07, g09, g10, g13). Compared with its previous version [6] (Table 2) this new diversity mechanism improved the quality of the results in problems g02, g05, g09 and g10. Also, the robustness of the results was better in problems g02, g05, g08, g09, g10 and g13.

Table 6. Comparison of our new version of the SES with respect to ASCHEA [10]. NA = Not Available. A result in **boldface** means a better value obtained by our new approach.

		Best Result		Mean Result		Worst Result	
Problem	**Optimal**	**New SES**	**ASCHEA**	**New SES**	**ASCHEA**	**New SES**	**ASCHEA**
g01	−15.0	−15.00	−15.0	**−15.00**	−14.84	−15.00	NA
g02	0.803619	**0.803601**	0.785	**0.785238**	0.59	0.751322	NA
g03	1.00	1.00	1.0	**1.00**	0.99989	1.00	NA
g04	−30665.539	−30665.539	30665.5	−30665.539	30665.5	−30665.539	NA
g05	5126.498	5126.599	5126.5	5174.492	5141.65	5304.167	NA
g06	−6961.814	−6961.814	−6961.81	−6961.284	−6961.81	−6952.482	NA
g07	24.306	**24.327**	24.3323	**24.475**	24.66	24.843	NA
g08	0.095825	0.095825	0.095825	0.095825	0.095825	0.095825	NA
g09	680.630	680.632	680.630	680.643	680.641	680.719	NA
g10	7049.25	**7051.90**	7061.13	7253.05	7193.11	7638.37	NA
g11	0.75	0.75	0.75	0.75	0.75	0.75	NA
g12	1.00	1.00	NA	1.00	NA	1.00	NA
g13	0.053950	0.053986	NA	0.166385	NA	0.468294	NA

When compared with respect to the three state-of-the-art techniques previously indicated, we found the following: Compared with the Homomorphous Maps (Table 3) the new SES found a better "best" solution in ten problems (g01, g02, g03, g04, g05, g06, g07, g09, g10 and g12) and a similar "best" result in other two (g08 and g11). Also, our technique reached better "mean" and "worst" results in ten problems (g01, g03, g04, g05, g06, g07, g08, g09, g10 and g12). A "similar" mean and worst result was found in problem g11. The Homomorphous maps found a "better" mean and worst result in function g02. No comparisons were made with respect to function g13 because such results were not available for HM.

With respect to Stochastic Ranking (Table 5), our approach was able to find a better "best" result in functions g02 and g10. In addition, it found a "similar" best solution in seven problems (g01, g03, g04, g06, g08, g11 and g12). Slightly better "best" results were found by SR in the remaining functions (g05, g07, g09 and g13). The new SES found better "mean" and "worst" results in four test functions (g02, g06, g09 and g10). It also provided similar "mean" and "worst" results in six functions (g01, g03, g04, g08, g11 and g12). Finally, SR found again just slightly better "mean" and "worst" results in functions g05, g07 and g13.

Compared against the Adaptive Segregational Constraint Handling Evolutionary Algorithm (Table 6), our algorithm found "better" best solutions in three problems (g02, g07 and g10) and it found "similar" best results in six functions (g01, g03, g04, g06, g08, g11). ASCHEA found slightly "better" best results in function g05 and g09. Additionally, the new SES found "better" mean results in four problems (g01, g02, g03 and g07) and it found "similar" mean results in three functions (g04, g08 and g11). ASCHEA surpassed our mean results in four functions (g05, g06, g09 and g10). We did not compare the worst results because they were not available for ASCHEA. We did not perform comparisons with respect to ASCHEA using functions g12 and g13 for the same reason. As we can see, our approach showed a very competitive performance with respect to these three state-of-the-art approaches.

Our approach can deal with moderately constrained problems (g04), highly constrained problems, problems with low (g06, g08), moderated (g09) and high (g01, g02, g03, g07) dimensionality, with different types of combined constraints (linear, nonlinear, equality and inequality) and with very large (g02), very small (g05 and g13) or even disjoint (g12) feasible regions. Also, the algorithm is able to deal with large search spaces (based on the intervals of the decision variables) with a very small feasible region (g10). Furthermore, the approach can find the global optimum in problems where such optimum lies on the boundaries of the feasible region (g01, g02, g04, g06, g07, g09). This behavior suggests that the mechanism of maintaining the best infeasible solution helps the search to sample the boundaries of the feasible region.

Besides still being a very simple approach, it is worth reminding that our algorithm does not require the fine-tuning of any extra parameters (other than those used with an evolution strategy) since the only parameters required by the approach have remained fixed in all cases. In contrast, the Homomorphous maps require an additional parameter (called v) which has to be found empirically [11]. Stochastic ranking requires the definition of a parameter called P_f, whose value has an important impact on the performance of the approach [9]. ASCHEA also requires the definition of several extra parameters, and in its latest version, it uses niching, which is a process that also has at least one additional parameter [10].

The computational cost measured in terms of the number of fitness function evaluations (FFE) performed by any approach is lower for our algorithm with respect to the others to respect to which it was compared. This is an additional (and important) advantage, mainly if we wish to use this approach for solving real-world problems. Our new approach performed $240,000$ FFE, the previous version required $330,000$ FFE, the Stochastic Ranking performed $350,000$ FFE, the Homomorphous Maps performed $1,400,000$ FFE, and ASCHEA required $1,500,000$ FFE.

6 Conclusions and Future Work

An improved diversity mechanism added to a multimembered Evolution Strategy combined with some selection criteria based on feasibility were proposed to solve (rather efficiently) constrained optimization problems. The proposed approach does not require the use of a penalty function and it does not require the fine-tuning of any extra parameters (other than those required by an evolution strategy), since they assume fixed values. The proposed approach uses the self-adaptation mechanism of a multimembered ES to sample the search space in order to reach the feasible region and it uses three simple selection criteria based on feasibility to guide the search towards the global optimum. Moreover, the proposed technique adopts a diversity mechanism which consists of allowing infeasible solutions close to the boundaries of the feasible region to remain in the next population. This approach is very easy to implement and its computational cost (measured in terms of the number of fitness function evaluations) is considerably lower than the cost reported by other three constraint-handling techniques which are representative of the state-of-the-art in evolutionary optimization. Despite its lower computational cost, the proposed approach was able to match (and even improve) on the results obtained by the other algorithms with respect to which it was compared.

As part of our future work, we plan to evaluate the rate at which our algorithm reaches the feasible region. This is an important issue when dealing with real-world applications, since in highly constrained search spaces, reaching the feasible region may be a rather costly task. Additionally, we have to perform more experiments in order to establish which of the three mechanisms of the approach (diversity mechanism, combined crossover or the reduced stepsize) is mandatory or if only their combined effect makes the algorithm work.

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Appendix: Test Functions

1. **g01**: Minimize: $f(\boldsymbol{x}) = 5\sum_{i=1}^{4} x_i - 5\sum_{i=1}^{4} x_i^2 - \sum_{i=5}^{13} x_i$ subject to:

$$g_1(\boldsymbol{x}) = 2x_1 + 2x_2 + x_{10} + x_{11} - 10 \le 0,\ \ g_2(\boldsymbol{x}) = 2x_1 + 2x_3 + x_{10} + x_{12} - 10 \le 0$$
$$g_3(\boldsymbol{x}) = 2x_2 + 2x_3 + x_{11} + x_{12} - 10 \le 0,\ \ g_4(\boldsymbol{x}) = -8x_1 + x_{10} \le 0,$$
$$g_5(\boldsymbol{x}) = -8x_2 + x_{11} \le 0,\ \ g_6(\boldsymbol{x}) = -8x_3 + x_{12} \le 0,\ \ g_7(\boldsymbol{x}) = -2x_4 - x_5 + x_{10} \le 0,$$
$$g_8(\boldsymbol{x}) = -2x_6 - x_7 + x_{11} \le 0,\ \ g_9(\boldsymbol{x}) = -2x_8 - x_9 + x_{12} \le 0$$

   where the bounds are $0 \le x_i \le 1$ ($i = 1, \ldots, 9$), $0 \le x_i \le 100$ ($i = 10, 11, 12$) and $0 \le x_{13} \le 1$. The global optimum is at $x^* = (1,1,1,1,1,1,1,1,1,3,3,3,1)$ where $f(x^*) = -15$. Constraints g_1, g_2, g_3, g_4, g_5 and g_6 are active.
2. **g02**: Maximize: $f(\boldsymbol{x}) = \left| \frac{\sum_{i=1}^{n} \cos^4(x_i) - 2\prod_{i=1}^{n} \cos^2(x_i)}{\sqrt{\sum_{i=1}^{n} i x_i^2}} \right|$ subject to:

$$g_1(\boldsymbol{x}) = 0.75 - \prod_{i=1}^{n} x_i \le 0,\ \ g_2(\boldsymbol{x}) = \sum_{i=1}^{n} x_i - 7.5n \le 0$$

   where $n = 20$ and $0 \le x_i \le 10$ ($i = 1, \ldots, n$). The global maximum is unknown; the best reported solution is [9] $f(x^*) = 0.803619$. Constraint g_1 is close to being active ($g_1 = -10^{-8}$).
3. **g03**: Maximize: $f(\boldsymbol{x}) = (\sqrt{n})^n \prod_{i=1}^{n} x_i$ subject to: $h(\boldsymbol{x}) = \sum_{i=1}^{n} x_i^2 - 1 = 0$ where $n = 10$ and $0 \le x_i \le 1$ ($i = 1, \ldots, n$). The global maximum is at $x_i^* = 1/\sqrt{n}$ ($i = 1, \ldots, n$) where $f(x^*) = 1$.
4. **g04**: Minimize: $f(\boldsymbol{x}) = 5.3578547x_3^2 + 0.8356891x_1x_5 + 37.293239x_1 - 40792.141$ subject to:
   $g_1(\boldsymbol{x}) = 85.334407 + 0.0056858x_2x_5 + 0.0006262x_1x_4 - 0.0022053x_3x_5 - 92 \le 0$
   $g_2(\boldsymbol{x}) = -85.334407 - 0.0056858x_2x_5 - 0.0006262x_1x_4 + 0.0022053x_3x_5 \le 0$
   $g_3(\boldsymbol{x}) = 80.51249 + 0.0071317x_2x_5 + 0.0029955x_1x_2 + 0.0021813x_3^2 - 110 \le 0$
   $g_4(\boldsymbol{x}) = -80.51249 - 0.0071317x_2x_5 - 0.0029955x_1x_2 - 0.0021813x_3^2 + 90 \le 0$
   $g_5(\boldsymbol{x}) = 9.300961 + 0.0047026x_3x_5 + 0.0012547x_1x_3 + 0.0019085x_3x_4 - 25 \le 0$
   $g_6(\boldsymbol{x}) = -9.300961 - 0.0047026x_3x_5 - 0.0012547x_1x_3 - 0.0019085x_3x_4 + 20 \le 0$
   where: $78 \le x_1 \le 102$, $33 \le x_2 \le 45$, $27 \le x_i \le 45$ ($i = 3, 4, 5$). The optimum solution is $x^* = (78, 33, 29.995256025682, 45, 36.775812905788)$ where $f(x^*) = -30665.539$. Constraints g_1 y g_6 are active.
5. **g05**: Minimize: $f(\boldsymbol{x}) = 3x_1 + 0.000001x_1^3 + 2x_2 + (0.000002/3)x_2^3$ subject to: $g_1(\boldsymbol{x}) = -x_4 + x_3 - 0.55 \le 0,\ \ g_2(\boldsymbol{x}) = -x_3 + x_4 - 0.55 \le 0$
   $h_3(\boldsymbol{x}) = 1000\sin(-x_3 - 0.25) + 1000\sin(-x_4 - 0.25) + 894.8 - x_1 = 0$
   $h_4(\boldsymbol{x}) = 1000\sin(x_3 - 0.25) + 1000\sin(x_3 - x_4 - 0.25) + 894.8 - x_2 = 0$
   $h_5(\boldsymbol{x}) = 1000\sin(x_4 - 0.25) + 1000\sin(x_4 - x_3 - 0.25) + 1294.8 = 0$ where $0 \le x_1 \le 1200$, $0 \le x_2 \le 1200$, $-0.55 \le x_3 \le 0.55$, and $-0.55 \le x_4 \le 0.55$. The best known solution is $x^* = (679.9453, 1026.067, 0.1188764, -0.3962336)$ where $f(x^*) = 5126.4981$.

6. **g06**: Minimize: $f(\boldsymbol{x}) = (x_1 - 10)^3 + (x_2 - 20)^3$ subject to: $g_1(\boldsymbol{x}) = -(x_1 - 5)^2 - (x_2 - 5)^2 + 100 \le 0$, $g_2(\boldsymbol{x}) = (x_1 - 6)^2 + (x_2 - 5)^2 - 82.81 \le 0$ where $13 \le x_1 \le 100$ and $0 \le x_2 \le 100$. The optimum solution is $x^* = (14.095, 0.84296)$ where $f(x^*) = -6961.81388$. Both constraints are active.
7. **g07**: Minimize: $f(\boldsymbol{x}) = x_1^2 + x_2^2 + x_1x_2 - 14x_1 - 16x_2 + (x_3 - 10)^2 + 4(x_4 - 5)^2 + (x_5 - 3)^2 + 2(x_6 - 1)^2 + 5x_7^2 + 7(x_8 - 11)^2 + 2(x_9 - 10)^2 + (x_{10} - 7)^2 + 45$
subject to: $g_1(\boldsymbol{x}) = -105 + 4x_1 + 5x_2 - 3x_7 + 9x_8 \le 0$
$g_2(\boldsymbol{x}) = 10x_1 - 8x_2 - 17x_7 + 2x_8 \le 0$, $g_3(\boldsymbol{x}) = -8x_1 + 2x_2 + 5x_9 - 2x_{10} - 12 \le 0$
$g_4(\boldsymbol{x}) = 3(x_1 - 2)^2 + 4(x_2 - 3)^2 + 2x_3^2 - 7x_4 - 120 \le 0$
$g_5(\boldsymbol{x}) = 5x_1^2 + 8x_2 + (x_3 - 6)^2 - 2x_4 - 40 \le 0$
$g_6(\boldsymbol{x}) = x_1^2 + 2(x_2 - 2)^2 - 2x_1x_2 + 14x_5 - 6x_6 \le 0$
$g_7(\boldsymbol{x}) = 0.5(x_1 - 8)^2 + 2(x_2 - 4)^2 + 3x_5^2 - x_6 - 30 \le 0$
$g_8(\boldsymbol{x}) = -3x_1 + 6x_2 + 12(x_9 - 8)^2 - 7x_{10} \le 0$ where $-10 \le x_i \le 10$ $(i = 1, \dots, 10)$. The global optimum is $x^* = (2.171996, 2.363683, 8.773926, 5.095984, 0.9906548, 1.430574, 1.321644, 9.828726, 8.280092, 8.375927)$ where $f(x^*) = 24.3062091$. Constraints g_1, g_2, g_3, g_4, g_5 and g_6 are active.
8. **g08**: Maximize: $f(\boldsymbol{x}) = \frac{\sin^3(2\pi x_1)\sin(2\pi x_2)}{x_1^3(x_1+x_2)}$
subject to: $g_1(\boldsymbol{x}) = x_1^2 - x_2 + 1 \le 0$, $g_2(\boldsymbol{x}) = 1 - x_1 + (x_2 - 4)^2 \le 0$ where $0 \le x_1 \le 10$ and $0 \le x_2 \le 10$. The optimum solution is located at $x^* = (1.2279713, 4.2453733)$ where $f(x^*) = 0.095825$.
9. **g09**: Minimize: $f(\boldsymbol{x}) = (x_1 - 10)^2 + 5(x_2 - 12)^2 + x_3^4 + 3(x_4 - 11)^2 + 10x_5^6 + 7x_6^2 + x_7^4 - 4x_6x_7 - 10x_6 - 8x_7$
subject to: $g_1(\boldsymbol{x}) = -127 + 2x_1^2 + 3x_2^4 + x_3 + 4x_4^2 + 5x_5 \le 0$, $g_2(\boldsymbol{x}) = -282 + 7x_1 + 3x_2 + 10x_3^2 + x_4 - x_5 \le 0$
$g_3(\boldsymbol{x}) = -196 + 23x_1 + x_2^2 + 6x_6^2 - 8x_7 \le 0$, $g_4(\boldsymbol{x}) = 4x_1^2 + x_2^2 - 3x_1x_2 + 2x_3^2 + 5x_6 - 11x_7 \le 0$ where $-10 \le x_i \le 10$ $(i = 1, \dots, 7)$. The global optimum is $x^* = (2.330499, 1.951372, -0.4775414, 4.365726, -0.6244870, 1.038131, 1.594227)$ where $f(x^*) = 680.6300573$. Two constraints are active (g_1 and g_4).
10. **g10**: Minimize: $f(\boldsymbol{x}) = x_1 + x_2 + x_3$ subject to: $g_1(\boldsymbol{x}) = -1 + 0.0025(x_4 + x_6) \le 0$
$g_2(\boldsymbol{x}) = -1 + 0.0025(x_5 + x_7 - x_4) \le 0$, $g_3(\boldsymbol{x}) = -1 + 0.01(x_8 - x_5) \le 0$
$g_4(\boldsymbol{x}) = -x_1x_6 + 833.33252x_4 + 100x_1 - 83333.333 \le 0$
$g_5(\boldsymbol{x}) = -x_2x_7 + 1250x_5 + x_2x_4 - 1250x_4 \le 0$
$g_6(\boldsymbol{x}) = -x_3x_8 + 1250000 + x_3x_5 - 2500x_5 \le 0$ where $100 \le x_1 \le 10000$, $1000 \le x_i \le 10000$, $(i = 2, 3)$, $10 \le x_i \le 1000$, $(i = 4, \dots, 8)$. The global optimum is: $x^* = (579.19, 1360.13, 5109.92, 182.0174, 295.5985, 217.9799, 286.40, 395.5979)$, where $f(x^*) = 7049.25$. g_1, g_2 and g_3 are active.
11. **g11**: Minimize: $f(\boldsymbol{x}) = x_1^2 + (x_2 - 1)^2$ subject to: $h(\boldsymbol{x}) = x_2 - x_1^2 = 0$ where: $-1 \le x_1 \le 1$, $-1 \le x_2 \le 1$. The optimum solution is $x^* = (\pm 1/\sqrt{2}, 1/2)$ where $f(x^*) = 0.75$.
12. **g12**: Maximize: $f(\boldsymbol{x}) = \frac{100 - (x_1-5)^2 - (x_2-5)^2 - (x_3-5)^2}{100}$ subject to: $g_1(\boldsymbol{x}) = (x_1 - p)^2 + (x_2 - q)^2 + (x_3 - r)^2 - 0.0625 \le 0$ where $0 \le x_i \le 10$ $(i = 1, 2, 3)$ and $p, q, r = 1, 2, \dots, 9$. The feasible region of the search space consists of 9^3 disjointed spheres. A point (x_1, x_2, x_3) is feasible if and only if there exist p, q, r such the above inequality (12) holds. The global optimum is located at $x^* = (5, 5, 5)$ where $f(x^*) = 1$.
13. **g13**: Minimize: $f(\boldsymbol{x}) = e^{x_1x_2x_3x_4x_5}$ subject to: $h_1(\boldsymbol{x}) = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 - 10 = 0$
$h_2(\boldsymbol{x}) = x_2x_3 - 5x_4x_5 = 0$, $h_3(\boldsymbol{x}) = x_1^3 + x_2^3 + 1 = 0$ where $-2.3 \le x_i \le 2.3$ $(i = 1, 2)$ and $-3.2 \le x_i \le 3.2$ $(i = 3, 4, 5)$. The optimum solution is $x^* = (-1.717143, 1.595709, 1.827247, -0.7636413, -0.763645)$ where $f(x^*) = 0.0539498$.